ISSN 2686-9667. Вестник российских университетов. Математика

Том 24, № 127 2019

DOI 10.20310/2686-9667-2019-24-127-252-271
УДК 512.643

Core of a matrix in max algebra and in nonnegative algebra: A survey

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Сердцевина матрицы в макс-алгебре и в неотрицательной алгебре: Обзор

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Abstract. This paper presents a light introduction to Perron–Frobenius theory in max algebra and in nonnegative linear algebra, and a survey of results on two cores of a nonnegative matrix. The (usual) core of a nonnegative matrix is defined as $\cap_{k\geqslant 1}\text{span}_+(A^k)$, that is, intersection of the nonnegative column spans of matrix powers. This object is of importance in the (usual) Perron-Frobenius theory, and it has some applications in ergodic theory. We develop the direct max-algebraic analogue and follow the similarities and differences of both theories.

Keywords: max algebra; nonnegative matrix theory; Perron–Frobenius theory; matrix power; eigenspace; core

Acknowledgements: The work is partially supported by EPSRC (project no. RRAH15735), S. Sergeev also acknowledges the support of RFBR-CNRS (project no. 11-0193106) and RFBR (project no. 12-01-00886).

For citation: Butkovič P., Schneider H., Sergeev S. Core of a matrix in max algebra and in nonnegative algebra: A survey. *Vestnik rossiyskikh universitetov. Matematika – Russian Universities Reports. Mathematics*, 2019, vol. 24, no. 127, pp. 252–271. DOI 10.20310/2686-9667-2019-24-127-252-271.

Аннотация. Эта статья предлагает краткое введение в теорию Перрона–Фробениуса в макс-алгебре и в неотрицательной линейной алгебре, а также обсуждение результатов, касающихся сердцевин неотрицательных матриц, понимаемых в двух смыслах. Обычная сердцевина неотрицательной матрицы определяется как $\cap_{k\geqslant 1}\text{span}_+(A^k)$, то есть как пересечение подпространств, натянутых на неотрицательные столбцы степеней этой матрицы. Этот объект важен для обычной теории Перрона–Фробениуса. Он имеет приложения в эргодической теории. Мы прослеживаем прямую макс-алгебраическую аналогию и проявляем совпадения и различия обеих теорий.

Ключевые слова: макс-алгебра; теория неорицательных матриц; теория Перрона–Фробениуса; степень матрицы; собственнное подпространство; сердцевина

Благодарности: Работа выполнена при поддержке EPSRC (проект № RRAH15735), С. Сергеев также благодарит за поддержку РФФИ-CNRS (проект № 11-0193106) и РФФИ (проект № 12-01-00886).

Для цитирования: *Буткович П., Шнайдер Г., Сергеев С.* Сердцевина матрицы в макс-алгебре и в неотрицательной алгебре: Обзор // Вестник российских университетов. Математика. 2019. Т. 24. № 127. С. 252–271. DOI 10.20310/2686-9667-2019-24-127-252-271. (In Engl., Abstr. in Russian)

1. Introduction

1.1. Basic Perron-Frobenius theory

We study the matrices with nonnegative entries, such as the following one:

$$\begin{pmatrix} 0 & 0 & 2 & 0 \\ 0.5 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0.1 \\ 0 & 0.2 & 0 & 1 \end{pmatrix}. \tag{1.1.1}$$

The set of matrices of dimension n with real nonnegative entries will be denoted by $\mathbb{R}_+^{n\times n}$.

With a matrix $A=(a_{ij})\in\mathbb{R}_+^{n\times n}$ we associate a weighted (di)graph $\mathcal{G}(A)$ with the set of nodes $N=\{1,\dots,n\}$ and set of edges $E\subseteq N\times N$ containing a pair (i,j) if and only if $a_{ij}\neq 0$; the weight of an edge $(i,j)\in E$ is defined to be $w(i,j):=a_{ij}$. A graph with just one node and no edge will be called *trivial*. The digraph associated with (1.1.1) is shown on Figure 1.

A path P in $\mathcal{G}(A)$ is a sequence of nodes $i_0,i_1,...,i_t$ such that each pair $(i_0,i_1),(i_1,i_2)$, ..., (i_{t-1},i_t) is an edge in $\mathcal{G}(A)$. It has *length* $l(P):=t$ and *weight* $w(P):=w(i_0,i_1)\cdot w(i_1,i_2)\cdots w(i_{t-1},i_t)$, and is called an $i-j$ path if $i_0=i$ and $i_t=j$. A path P is called a *cycle* if $i_0=i_t$, and a cycle is called *elementary* if all nodes of the cycle are different. In particular, consider the following elementary cycles on the digraph on Figure 1: " 3, 2, 1, 3 ", " 3, 4, 2, 1, 3 ", " 4, 4 "

$A=(a_{ij})\in\mathbb{R}_+^{n\times n}$ is irreducible if $\mathcal{G}(A)$ is trivial or for any $i,j\in\{1,\dots,n\}$ there is an $i-j$ path. Otherwise A is reducible.

We do not actually list all the cycles here. As defined above, two cycles may have the same set of edges but different start and end nodes.

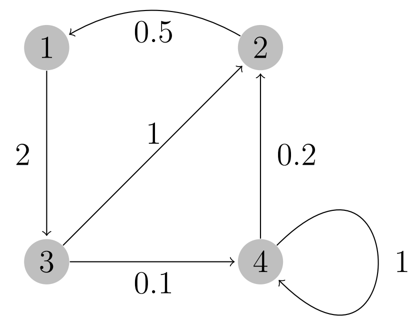

Figure 1: The weighted digraph associated with (1.1.1).

Observe that (1.1.1) is irreducible. For instance, both $3, 2, 1$ and $3, 4, 2, 1$ are $3{-}1$ paths.

By the Perron–Frobenius theorem, any irreducible matrix $A \in \mathbb{R}_+^{n\times n}$ has a positive eigenvalue, which is of the largest modulus among all the eigenvalues of A. This eigenvalue ρ is simple, that is, all eigenvectors associated with it are multiples of just one eigenvector (nonzero x satisfying $Ax = \rho x$). This eigenvalue is called the Perron root of A and denoted by $\rho^+(A)$.

Notation $A^{\times k}$ will stand for the usual kth power of a nonnegative matrix.

An irreducible nonnegative matrix does not have any other eigenvalues with a nonnegative eigenvector. Indeed, let $\rho := \rho^+(A)$, and let λ be such an eigenvalue, then $\lambda < \rho$. Let x, resp. y be two nonzero eigenvectors associated with ρ, resp. λ. The irreducibility of A then implies that both x and y have all components positive, and then there exists a number s such that $x \leqslant sy$. Then we also have $A^{\times t}x \leqslant sA^{\times t}y$ for all t, so $\rho^t x \leqslant s\lambda^t y$ for all t. However, if $\rho > \lambda$, then this is impossible.

In the case of (1.1.1), applying the MATLAB function "eig (A)" we can find that the four eigenvalues (over complex field) are equal, approxiamately, to $-0.4966 + 0.8641i$, $-0.4966 - 0.8641i$, 1.0785 and 0.9147. The first two eigenvalues are complex conjugates of each other, with the absolute value (approximately) 0.9967, and the corresponding eigenvectors are also complex. The last two eigenvalues are real positive, but only the bigger eigenvalue 1.0785 has a positive eigenvector: approximately $(0.5873\ 0.2723\ 0.3167\ 0.6933)$. So 1.0785 is the Perron root of (1.1.1), with essentially unique (real positive) Perron eigenvector.

In the general (reducible) case, a matrix A may have several eigenvalues with nonnegative eigenvectors, but in general, not all eigenvalues of A have this property. The structure of the set of eigenvectors associated with a particular eigenvalue may be also not so trivial. The set of nonnegative eigenvectors associated with a particular eigenvalue ρ is denoted by $V_+(A, \rho)$, and it is a convex cone.

Recall that a set $V \subseteq \mathbb{R}_+^n$ is called a *convex cone* if 1) $\alpha v \in V$ for all $v \in V$ and $\alpha \in \mathbb{R}_+$, 2) $u + v \in V$ for $u, v \in V$. Convex sets and convex polytopes can be viewed as section of convex cones by planes (for instance, requiring some coordinate to be constant). A convex cone V is said to be *generated* by $S \subseteq \mathbb{R}_+^n$ if each $v \in V$ can be represented as

The reasons for this unusual notation for (usual) matrix power will soon become clear a nonnegative linear combination $v = \bigoplus_{x \in S} \alpha_x x$ where only finitely many nonnegative α_x are different from zero. When V is generated by the columns of a matrix A, this is denoted by $V = \text{span}_+(A)$.

The Perron-Frobenius theorem and its extensions have many different proofs and applications. There are well-known applications in mathematical biology, say, in population dynamics [34], and most recently and notably, to Google PageRank. See Wikipedia, an original work of Frobenius [26], the survey of Schneider [45] and the textbooks of Berman-Plemmons [12] and Brualdi-Ryser [14].

Note that in what follows we are concerned only with nonnegative eigenvalues and nonnegative eigenvectors of a nonnegative matrix. In order to bring our terminology into line with the corresponding theory for max algebra we use the terms eigenvalue and eigenvector in a restrictive fashion. That is, we shall further call ρ an *eigenvalue* of a nonnegative matrix A (only) if there is a nonnegative eigenvector x of A for ρ. Further x will be called an *eigenvector* (only) if it is nonnegative.

1.2. Max-algebraic Perron-Frobenius

By max algebra we understand the set of nonnegative numbers $\mathbb{R}_+$ where the role of addition is played by taking maximum of two numbers: $a \oplus b := \max(a, b)$, and the multiplication is as in the usual arithmetics. This is carried over to matrices and vectors like in the usual linear algebra so that for two matrices $A = (a_{ij})$ and $B = (b_{ij})$ of appropriate sizes, $(A \oplus B)_{ij} = a_{ij} \oplus b_{ij}$ and $(A \otimes B)_{ik} = \bigoplus_k a_{ik} b_{kj}$. Notation $A^{\otimes k}$ will stand for the kth max-algebraic power.

In particular, we have $2 \times 2 = 4$ but $2 \oplus 2 = 2$, and

$$\begin{pmatrix} 1 & 3 \\ 5 & 6 \end{pmatrix} \otimes \begin{pmatrix} 3 & 5 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 5 \\ 15 & 25 \end{pmatrix}.$$

A set $V \subseteq \mathbb{R}_+^n$ will be called a *max cone* if 1) $\alpha v \in V$ for all $v \in V$ and $\alpha \in \mathbb{R}_+$, 2) $u \oplus v \in V$ for $u, v \in V$. Max cones are a special case of idempotent semimodules, see [8, 35]. A max cone V is said to be *generated* by $S \subseteq \mathbb{R}_+^n$ if each $v \in V$ can be represented as a max combination $v = \bigoplus_{x \in S} \alpha_x x$ where only finitely many (nonnegative) α_x are different from zero. When V is generated by the columns of a matrix A, this is denoted $V = \text{span}_\oplus(A)$. Max cones are max-algebraic analogues of convex cones.

A vector z in a max cone $V \subseteq \mathbb{R}_+^n$ is called an *extremal* if $z = u \oplus v$ and $u, v \in V$ imply $z = u$ or $z = v$. Any finitely generated max cone is generated by its extremals, see Wagneur [53], and [18, 28] for more recent extensions (for instance, the tropical Minkowski theorem). The *maximum cycle geometric mean* of A is defined by

$$\rho^\oplus(A) = \max\{w(C)^{1/l(C)};\ C \text{ is a cycle in } \mathcal{G}(A)\} \ . \qquad (1.2.2)$$

Recall that $w(C)$ denotes the product of all weights of the edges in C, and $l(C)$ is the number of edges (that is, the length). The *critical graph* of A, denoted by $\mathcal{C}(A)$, consists

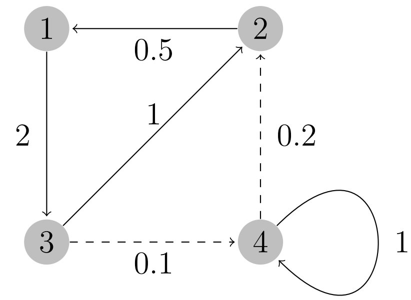

Figure 2: The critical graph of matrix (1.1.1).

of all nodes and edges belonging to the cycles which attain the maximum in (1.2.2). The set of such nodes will be called *critical* and denoted by N_c; the set of such edges will be called *critical* and denoted by E_c. Observe that the critical graph, defined as above, consists of several strongly connected subgraphs of $\mathcal{G}(A)$. Maximal such subgraphs are the *strongly connected components* of $\mathcal{C}(A)$. For example, the critical graph of (1.1.1) is shown on Figure 2 (the bold arcs).

If for $A \in \mathbb{R}_+^{n\times n}$ we have $A \otimes x = \rho x$ with $\rho \in \mathbb{R}_+$ and a nonzero $x \in \mathbb{R}_+^n$, then ρ is a *max(-algebraic) eigenvalue* and x is a *max(-algebraic) eigenvector* associated with ρ. The set of max eigenvectors x associated with ρ, with the zero vector adjoined to it, is a max cone further denoted by $V_\oplus(A,\rho)$. It is called the *eigencone* of A associated with ρ.

In general (reducible) case, a matrix $A \in \mathbb{R}_+^{n\times n}$ may have several max eigenvalues. The greatest max eigenvalue is equal to $\rho^\oplus(A)$ (see [11,16,24,32]), and it is called the *principal eigenvalue*. The corresponding eigencone is called the *principal eigencone*. It is also known that if A is irreducible then $\rho^\oplus(A)$ is the only eigenvalue, which we call the *max-(algebraic) Perron root* of A (the proof of this uniqueness is the same as in the classical argument writen above for the nonnegative case). However, unlike in the usual nonnegative Perron-Frobenius theory discussed above, an irreducible matrix A may have several eigenvectors. For instance, (1.1.1) has the following non-proportional eigenvectors:

$$v^{(1)} = (1\ 0.5\ 0.5\ 0.1),\quad v^{(2)} = (0.2\ 0.1\ 0.1\ 1).$$

Roughly speaking, extremal vectors of $V_\oplus(A,\rho^\oplus(A))$ correspond to the components of the critical graph. For explicit description of $V_\oplus(A,\rho^\oplus(A))$, see Theorem 3.4.2 below. It uses the *Kleene star*

$$A^* = I \oplus A \oplus A^2 \oplus A^3 \oplus \dots, \tag{1.2.3}$$

where I denotes the identity matrix. Series (1.2.3) converges if and only if $\rho^\oplus(A) \leqslant 1$, in which case $A^* = I \oplus A \oplus \ldots \oplus A^{n-1}$. Note that if $\rho^\oplus(A) \neq 0$, then $\rho^\oplus(A/\rho^\oplus(A)) = 1$, and so $(A/\rho^\oplus(A))^*$ always converges.

The *path interpretation* of max-algebraic matrix powers $A^{\otimes l}$ is that each entry $a_{ij}^{\otimes l}$ is equal to the greatest weight of $i - j$ paths with length l. Consequently, for $i \neq j$, the entry a^*_{ij} of A^* is equal to the greatest weight of $i - j$ paths (with no length restrictions).

1.3. Max algebra: historical notes

The max-algebraic eigenproblem is perhaps one of the most well-known efficiently resolved problems of max algebra. Its theory was initiated, in particular, by Cuninghame–Green [24, 25] and Vorobyev [1–3], with scheduling and economic motivations in mind. The full description of eigenvector cone in the irreducible case was written by Gondran and Minoux [30], and the reducible case was described by Gaubert [27], see also [17] for a complete exposition. Further evolution of max algebra and its applications in scheduling and discrete event systems can be learnt from [5, 11, 32]. In Russia, max algebra was developed by academician Maslov and his school [7, 8, 39] as algebraic foundation of *idempotent analysis*, a new area of mathematics with applications in mathematical physics and optimal control. In particular, Dudnikov and Samborkiĭ [4] and later Shpiz [9] extended the max-algebraic eigenvector existence theorem to more general idempotent semimodules. Litvinov, Maslov and Sobolevskiĭ [6] developed idempotent interval analysis. For the current developments in max algebra, idempotent analysis, tropical convexity and related areas, see, e.g., survey of Akian, Bapat and Gaubert [10], monographs of Butkovič [16] and McEneaney [40], collections of papers [36–38].

1.4. Core of nonnegative matrix

The main topic of this paper, which is mostly a shorter version of [19], is the core of nonnegative matrix, defined in nonnegative algebra as $\text{core}_+(A) := \cap_{k\geqslant 1}\text{span}_+(A^{\times k})$, and in max algebra as, $\text{core}_\oplus(A) := \cap_{k\geqslant 1}\,\text{span}_\oplus\,(A^{\otimes k})$ (so that we can write

$$\text{core}(A) := \cap_{k\geqslant 1}\,\text{span}(A^k)$$

to unite both definitions). The concept of matrix core was introduced by Pullman in [44]. This led to a geometric approach to the proof of the Perron-Frobenius theorem based on the properties of the core. Pullman investigated the action of a matrix on its core showing that it is bijective and that the extremal rays of the core can be partitioned into periodic orbits. In other words, extremal rays of the core of A are nonnegative eigenvectors of the powers of A (associated with positive eigenvalues).

Our main purpose in [19] was to extend Pullman's core to max algebra, thereby investigating the periodic sequence of eigencones of max-algebraic matrix powers. However, following the line of [18, 21, 33], we developed the theory in max algebra and nonnegative algebra simultaneously, in order to emphasize common features as well as differences, to provide general (simultaneous) proofs where this is possible. We did not aim to obtain new results on the usual core of a nonnegative matrix with respect to [44, 52] (although our unifying approach possibly led to new and more elementary proofs). Our motivation is closely related to the Litvinov-Maslov correspondence principle [35], viewing the idempotent mathematics (in particular, max algebra) as a "shadow" of the "traditional" mathematics over real and complex fields.

Pullman's core can be also seen as closely related to the limits of powers of nonnegative matrices. However it is a different concept. Consider the simple example

$$\begin{pmatrix} 1 & 0 \\ 0 & 0.5 \end{pmatrix}.$$

Then, for any nonnegative x, $A^k x$ will tend to a multiple of $(1,\ 0)^T$ while the core of A is the entire nonnegative orthant $\mathbb{R}^2_+$.

To the authors' knowledge, the core of a nonnegative matrix has not received much attention in linear algebra. However, a more detailed study has been carried out by Tam and Schneider [52], who extended the concept of core to linear mappings preserving a proper cone. The case when the core is a polyhedral (i. e., finitely generated) cone was examined in detail in [52, Section 3], and the results were applied to study the case of nonnegative matrix in [52, Section 4]. This work has found further applications in the theory of dynamic systems acting on the path space of a stationary Bratteli diagram. In particular, Bezuglyi et al. [13] describe and exploit a natural correspondence between ergodic measures and extremals of the core of the incidence matrix of such a diagram. The perspectives of a max-algebraic analogue of this theory are yet to be explored.

There is also much more literature on the related but distinct question of the limiting sets of homogeneous and non-homogeneous Markov chains in nonnegative algebra; see the books by Hartfiel [31] and Seneta [47] and, e.g., the works of Chi [22] and Sierksma [51]. In max algebra, see the results on the ultimate column span of matrix powers for irreducible matrices [16, Theorem 8.3.11], [48], and by Merlet [41] on the invariant max cone of non-homogeneous matrix products.

1.5. Organization

The rest of the paper is divided into two main sections: Preliminaries and Main results. Preliminaries are occupied with the rest of prerequisites, to understand the situation even better. The proofs of Main results can be found in [19]. In some cases, some hints for the proofs are given. Examples illustrating our results can be found in [19] (the last section).

2. Preliminaries

2.1. Ultimate periodicity and immediate periodicity

For a strongly connected graph $\mathcal{G}$, define its *cyclicity* σ as the gcd (greatest common divisor) of the lengths of all elementary cycles and the cyclicity of a trivial graph to be 1. For a (general) graph containing several maximal strongly connected components (such as the critical graph $\mathcal{C}(A)$), cyclicity is defined as the lcm of the cyclicities of the strongly connected components. A graph with cyclicity 1 is called *primitive*. The following result demonstrates importance of cyclicity of critical graph in max algebra. See also [11].

Theorem 2.1.1. *[23, Cyclicity Theorem, Cohen et al.]. Let $A \in \mathbb{R}_+^{n\times n}$ be irreducible and let σ be the cyclicity of $\mathcal{C}(A)$. Then σ is the smallest p such that there exists $T(A)$ with $A^{\otimes(t+p)} = (\rho^{\oplus})^p(A)A^{\otimes t}$ for all $t \geqslant T(A)$.*

In the case when A is (1.1.1), we have $T(A)=5$, and $A^{\otimes t}$, for $t=5,6,7$ are shown below:

$$\begin{pmatrix} 0.02 & 2 & 0.04 & 0.2 \\ 0.01 & 0.02 & 1 & 0.1 \\ 0.5 & 0.02 & 0.02 & 0.1 \\ 0.1 & 0.2 & 0.2 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0.04 & 0.04 & 0.2 \\ 0.01 & 1 & 0.02 & 0.1 \\ 0.01 & 0.02 & 1 & 0.1 \\ 0.1 & 0.2 & 0.2 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0.02 & 0.04 & 2 & 0.2 \\ 0.5 & 0.02 & 0.02 & 0.1 \\ 0.01 & 1 & 0.02 & 0.1 \\ 0.1 & 0.2 & 0.2 & 1 \end{pmatrix}.$$

Theorem 2.1.1 is closely related to the theory of graph exponents as presented, for instance, in Brualdi-Ryser [14]. We will need the following formal definition.

A sequence $\{\aleph_k\}_{k\geqslant 1}$ is called *periodic* if there exists an integer p such that $\aleph_{k+p}$ is identical with $\aleph_k$ for all k. The least such p is called the *period* of $\{\aleph_k\}_{k\geqslant 1}$. A sequence $\{\aleph_k\}_{k\geqslant 1}$ is called *ultimately periodic* if the sequence $\{\aleph_k\}_{k\geqslant T}$ is periodic for some $T\geqslant 1$. The least such T is called the *periodicity threshold* of $\{\aleph_k\}_{k\geqslant 1}$.

In terms of the ultimate periodicity, Theorem 2.1.1 can be formulated as follows: for any irreducible nonnegative matrix $A\in\mathbb{R}_+^{n\times n}$, the sequence of matrix powers $\{(A/\rho^{\oplus}(A))^{\otimes t}\}$ with $t\geqslant 1$ is ultimately periodic with the period equal to the cyclicity of critical graph. Slightly generalizing the notion of ultimate periodicity, it can be also said that $\{A^{\otimes t}\}$ with $t\geqslant 1$ is ultimately periodic with growth rate $\rho^{\oplus}(A)$.

We also note that for a general reducible matrix $A\in\mathbb{R}_+^{n\times n}$, not all the sequences $\{a_{ij}^{(t)}\}_{t\geqslant 1}$ for $i,j\in\{1,\ldots,n\}$, are ultimately periodic in the sense of the definition given above. Such sequences can be decomposed into ultimately periodic subsequences with different growth rates, and the reader is referred to De Schutter [46], Gavalec [29] and Molnárová [42], Sergeev-Schneider [49] for more details.

2.2. Frobenius normal form

Every matrix $A=(a_{ij})\in\mathbb{R}_+^{n\times n}$ can be transformed by simultaneous permutations of the rows and columns in almost $O(n\log n)$ time to a *Frobenius Normal Form* (FNF) [12,14]

$$\begin{pmatrix} A_{11} & 0 & \ldots & 0 \\ A_{21} & A_{22} & \ldots & 0 \\ \ldots & \ldots & A_{\mu\mu} & \ldots \\ A_{r1} & A_{r2} & \ldots & A_{rr} \end{pmatrix}, \tag{2.2.1}$$

where $A_{11},\ldots,A_{rr}$ are irreducible square submatrices of A. They correspond to the sets of nodes $N_1,\ldots,N_r$ of the strongly connected components of $\mathcal{G}(A)$. Note that in (2.2.1) an edge from a node of N_μ to a node of N_ν in $\mathcal{G}(A)$ may exist only if $\mu\geqslant\nu$.

Generally, A_{KL} denotes the submatrix of A extracted from the rows with indices in $K\subseteq N$ and columns with indices in $L\subseteq N$, and $A_{\mu\nu}$ is a shorthand for $A_{N_\mu N_\nu}$.

Frobenius normal form of a matrix is not uniquely defined. Here is an example of a nonnegative matrix (left) and its Frobenius form (right)

$$\begin{pmatrix} 4 & 1 & 0 & 6 \\ 0 & 1 & 3 & 0 \\ 0 & 2 & 1 & 0 \\ 5 & 0 & 1 & 2 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 2 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 6 \\ 1 & 0 & 5 & 2 \end{pmatrix}.$$

If A is in the Frobenius Normal Form (2.2.1) then the *reduced graph*, denoted $R(A)$, is the (di)graph whose nodes correspond to N_μ, for $\mu = 1,\ldots,r$, and the set of arcs is $\{(\mu,\nu); (\exists k \in N_\mu)(\exists \ell \in N_\nu) a_{k\ell} > 0\}$. In max algebra and in nonnegative algebra, the nodes of $R(A)$ are *marked* by the corresponding eigenvalues (Perron roots), denoted by $\rho_\mu^\oplus := \rho^\oplus(A_{\mu\mu})$ (max algebra), $\rho_\mu^+ := \rho^+(A_{\mu\mu})$ (nonnegative algebra), and by ρ_μ when both algebras are considered simultaneously.

A class μ is trivial if $A_{\mu\mu}$ is the 1×1 zero matrix. Class μ *accesses* class ν, denoted $\mu\to\nu$, if $\mu=\nu$ or if there exists a $\mu\to\nu$ path in $R(A)$. A class is called *initial*, resp. *final*, if it is not accessed by, resp. if it does not access, any other class. Node i of $\mathcal{G}(A)$ accesses class ν, denoted by $i\to\nu$, if i belongs to a class μ such that $\mu\to\nu$.

2.3. Elements of reducible spectral theory

In this section we recall some elements of the spectral theory of reducible matrices in max algebra and in nonnegative linear algebra. All results are standard: the nonnegative part goes back to Frobenius [26], Sect. 11, and the max-algebraic counterpart is due to Gaubert [27], Ch. IV (see [16] for other references).

A class ν of A is called a *spectral class* of A associated with eigenvalue $\rho\neq 0$, or sometimes (A,ρ)-spectral class for short, if

$$\begin{aligned} &\rho_\nu^\oplus = \rho, \text{ and } \mu\to\nu \text{ implies } \rho_\mu^\oplus \leqslant \rho_\nu^\oplus \text{ (max algebra)},\\ &\rho_\nu^+ = \rho, \text{ and } \mu\to\nu, \mu\neq\nu \text{ implies } \rho_\mu^+ < \rho_\nu^+ \text{ (nonnegative algebra)}. \end{aligned} \tag{2.3.2}$$

In both algebras, note that there may be several spectral classes associated with the same eigenvalue. In nonnegative algebra, spectral classes are known as *distinguished classes* [45],

Denote by $\Lambda_+(A)$, resp. $\Lambda_\oplus(A)$, the set of **nonzero** eigenvalues of $A\in\mathbb{R}_+^{n\times n}$ in nonnegative linear algebra, resp. in max algebra. It will be denoted by $\Lambda(A)$ when both algebras are considered simultaneously, as in the following standard description.

Theorem 2.3.1. *[16, Th. 4.5.4], [45, Th. 3.7]. Let $A\in\mathbb{R}_+^{n\times n}$. Then*

$$\Lambda(A) = \{\rho_\nu \neq 0;\ \nu \textit{ is spectral}\}.$$

Theorem 2.3.1 encodes the following **two** statements:

$$\Lambda_\oplus(A) = \{\rho_\nu^\oplus \neq 0;\ \nu \text{ is spectral}\},\quad \Lambda_+(A) = \{\rho_\nu^+ \neq 0;\ \nu \text{ is spectral}\}, \tag{2.3.3}$$

where the notion of spectral class is defined in two different ways by (2.3.2), in two algebras.

See the illustrations of spectral classes of marked reduced graph in [16,17] (in max algebra) and in [21] (also in nonnegative algebra).

In both algebras, for each $\rho \in \Lambda(A)$ define

$$\begin{aligned} &A_\rho := \rho^{-1} \begin{pmatrix} 0 & 0 \\ 0 & A_{M_\rho M_\rho} \end{pmatrix}, \text{ where} \\ &M_\rho := \{i;\ i \to \nu,\ \nu \text{ is spectral with Perron root } \rho\} \ . \end{aligned} \tag{2.3.4}$$

Then the case of an eigencone associated with any eigenvalue can be reduced to the case of principal eigenvalue, as follows:

P r o p o s i t i o n 2.3.1. *[16, 27]. For $A \in \mathbb{R}_+^{n\times n}$ and each $\rho \in \Lambda(A)$, we have $V(A,\rho) = V(A_\rho, 1)$, where 1 is the principal eigenvalue of A_ρ.*

2.4. Access relations in matrix powers

In [19] we demonstrated that access relations and spectral classes of all matrix powers are similar, and that the case of an arbitrary eigenvalue reduces to the case of the principal eigenvalue. These results have simultaneous proofs in both algebras, which is due to the fact that the definitions of spectral classes are similar and that the associated **unweighted** digraphs of $A^{\times t}$ and $A^{\otimes t}$ are the same. Their nonnegative part goes back to Frobenius [26], and in some cases, is explicitly formulated in Tam-Schneider [52].

Lemma 2.4.1. *[19], [26, 52]. Let A be irreducible with the (unique) eigenvalue ρ, let $\mathcal{G}(A)$ have cyclicity σ and t be a positive integer. Then, A^t is a direct sum of gcd(t,σ) irreducible blocks with eigenvalues ρ^t, and A^t does not have eigenvalues other than ρ^t. The cyclicity of each block is $\sigma/\,$gcd(t,σ). In particular, all blocks of A^σ are primitive.*

Recall that each class μ of A corresponds to an irreducible submatrix $A_{\mu\mu}$. It is easy to see that $(A^t)_{\mu\mu} = (A_{\mu\mu})^t$ for any positive integer t. Suppose that the cyclicity of $\mathcal{G}(A_{\mu\mu})$ is σ. Applying Lemma 2.4.1 to $A_{\mu\mu}$ we see that μ gives rise to gcd(t,σ) classes in A^t, which are said to be *derived* from their common *ancestor* μ. The classes of A^t and A^l derived from the common ancestor will be called *related*. Note that this is an equivalence relation on the set of classes of all powers of A.

It can be checked that the same notions can be defined for the components of critical graphs, see [19].

Let us recall the following results on the similarity of access relations in matrix powers.

Lemma 2.4.2. *[19], [26, 52]. For all $t, l \geqslant 1$ and $\rho > 0$, an index $i \in \{1,\dots,n\}$ accesses (resp. is accessed by) a class with Perron root ρ^t in A^t if and only if it accesses (resp. is accessed by) a related class with Perron root ρ^l in A^l.*

Similar results holds for the strongly connected components of the critical graphs of matrix powers [19].

All eigenvalues and spectral classes of matrix powers are derived from those of A.

Theorem 2.4.1. *[19], [26, 52]. Let* $A \in \mathbb{R}_+^{n\times n}$ *and* $t \geqslant 1$.

(i) $\Lambda(A^t) = \{\rho^t;\ \rho \in \Lambda(A)\}$.

(ii) For each spectral class μ *of* A *with cyclicity* σ *there are* $gcd(t,\sigma)$ *spectral classes of* A^t *derived from it. Conversely, each spectral class of* A^t *is derived from a spectral class of* A.

As in the case of eigencones of a matrix, when working with $V(A^t, \rho^t)$ we can assume that $\rho = 1$ is the principal eigenvalue of A, and hence of all A^t.

Theorem 2.4.2. *[19], [26, 52]. Let* $A \in \mathbb{R}_+^{n\times n}$, $t \geqslant 1$ *and* $\rho \in \Lambda(A)$.

(i) $(A^t)_{M_\rho M_\rho} = (\rho^t (A_\rho)^t)_{M_\rho M_\rho}$.

(ii) $V(A^t, \rho^t) = V((A_\rho)^t, 1)$.

3. Core and eigencones

3.1. The main concepts

The notions given in this subsection are the central notions of [19]. They are defined in two algebras simultaneously.

Once again, the *core* of a nonnegative matrix A is defined as the intersection of the column spans (in other words, images) of its powers:

$$\text{core}(A) := \cap_{i=1}^{\infty} \text{span}(A^i). \tag{3.1.1}$$

The *(Minkowski) sum of eigencones* of a nonnegative matrix A is the cone consisting of all sums of vectors in all $V(A,\rho)$:

$$V^{\Sigma}(A) := \sum_{\rho\in\Lambda(A)} V(A,\rho). \tag{3.1.2}$$

If $\Lambda(A) = \emptyset$, which happens when $\rho(A) = 0$, then we assume that the sum on the right-hand side is $\{0\}$.

Further, the following notations can be seen as the "global" definition of cyclicity in nonnegative algebra and in max algebra.

1. σ_ρ is the the lcm of all cyclicities of spectral classes associated with $\rho \in \Lambda_+(A)$ (**nonnegative algebra**), or the cyclicity of critical graph associated with $\rho \in \Lambda_\oplus(A)$ (**max algebra**).

2. σ_Λ is the lcm of all σ_ρ where $\rho \in \Lambda(A)$.

3.2. Two cores of a nonnegative matrix

One of our main results relates the core with the sum of eigencones. The nonnegative part of this result can be found in Tam-Schneider [52, Th. 4.2, part (iii)], and the proof of it (in the nonnegative case) goes back to Pullman [44].

Theorem 3.2.1. *[19], [44, 52]. Let* $A \in \mathbb{R}_+^{n\times n}$. *Then*

$$\operatorname{core}(A) = \sum_{k\geqslant 1, \rho\in\Lambda(A)} V(A^k, \rho^k) = V^\Sigma(A^{\sigma_\Lambda}).$$

The following observations were used in the proof, and they are also of independent interest. They hold in both algebras with simultaneous proofs where only elementary analytic arguments are used.

P r o p o s i t i o n 3.2.1. *[19], [44]. Assume that* $\{K_l\}$ *for* $l \geqslant 1$, *is a sequence of cones in* $\mathbb{R}_+^n$ *such that* $K_{l+1} \subseteq K_l$ *for all* l, *and each of them generated by no more than* k *nonzero vectors. Then the intersection* $K = \cap_{l=1}^{\infty} K_l$ *is also generated by no more than* k *vectors.*

Proposition 3.2.1 seems to be an interesting geometric observation, which could be applied in a more general situation (for instance, in the context of tropical or nonnegative matrix semigroups).

P r o p o s i t i o n 3.2.2. *[19], [44]. Let* $A \in \mathbb{R}_+^{n\times n}$, *then*

(i) $\operatorname{core}(A)$ *is generated by no more than* n *vectors,*
(ii) the mapping induced by A *on* $\operatorname{core}(A)$ *is a surjection,*
(iii) the mapping induced by A *on the scaled extremals of* $\operatorname{core}(A)$ *is a permutation (i.e., a bijection).*

In the case of nonnegative algebra, the action of matrix on its core is not only surjective but also bijective. However, this does not hold in the case of max algebra, which leads us to the problem statements and results of [20].

3.3. Periodicity of the eigencone sequence

The following main result was obtained both in max and nonnegative algebra (Explicit publication of the (usual) nonnegative part of this result is unknown to us).

Theorem 3.3.1. *[19]. Let* $A \in \mathbb{R}_+^{n\times n}$. *Then*

(i) σ_ρ, *for* $\rho \in \Lambda(A)$, *is the period of the sequence* $\{V(A^k, \rho^k)\}_{k\geqslant 1}$, *and* $V(A^k, \rho^k) \subseteq V(A^{\sigma_\rho}, \rho^{\sigma_\rho})$ *for all* $k \geqslant 1$;
(ii) σ_Λ *is the period of the sequence* $\{V^\Sigma(A^k)\}_{k\geqslant 1}$, *and* $V^\Sigma(A^k) \subseteq V^\Sigma(A^{\sigma_\Lambda})$ *for all* $k \geqslant 1$.

More precise results can be formulated in the form of equivalence.

Theorem 3.3.2. *[19]. Let* $A \in \mathbb{R}_+^{n\times n}$ *and* σ *be either the cyclicities of spectral classes of* A **(nonnegative algebra)** *or the cyclicities of critical components of* A **(max algebra)**. *The following are equivalent for all positive* k, l:

(i) *gcd*(k,σ) *divides gcd*(l,σ) *for all cyclicities* σ;
(ii) *gcd*(k,σ_ρ) *divides gcd*(l,σ_ρ) *for all* $\rho \in \Lambda(A)$;

(iii) *gcd* (k, σ_Λ) *divides gcd* (l, σ_Λ) *;*

(iv) $V(A^k, \rho^k) \subseteq V(A^l, \rho^l)$ *for all* $\rho \in \Lambda(A)$ *and*

(v) $V^\Sigma(A^k) \subseteq V^\Sigma(A^l)$ *.*

Theorem 3.3.3. *Let* $A \in \mathbb{R}_+^{n\times n}$ *and* σ *be either the cyclicities of spectral classes (***nonnegative algebra***) or the cyclicities of critical components (***max algebra***) associated with some* $\rho \in \Lambda(A)$*. The following are equivalent for all positive* k, l*:*

(i) *gcd* (k, σ) *divides gcd* (l, σ) *for all cyclicities* σ *;*

(ii) *gcd* (k, σ_ρ) *divides gcd* (l, σ_ρ) *;*

(iii) $V(A^k, \rho^k) \subseteq V(A^l, \rho^l)$ *.*

The proof of Theorems 3.3.2 and 3.3.3 (and hence, Theorem 3.3.1 which can be obtained as their corollary) are based on the so-called Frobenius-Victory theorems, which are written, for both algebras, in the next subsection.

3.4. Perron-Frobenius and description of extremals

We now describe the principal eigencones in nonnegative linear algebra and then in max algebra. By means of Proposition 2.3.1, this description can be obviously extended to the general case. As in Section 2.3., both descriptions are essentially known: see [16, 26, 27, 45].

We emphasize that the vectors $x^{(\mu)}$ and $x^{(\widetilde{\mu})}$ appearing below are **full-size**.

Theorem 3.4.1. *Frobenius-Victory [45, Th. 3.7] Let* $A \in \mathbb{R}_+^{n\times n}$ *have* $\rho^+(A) = 1$ *.*

(i) *Each spectral class* μ *with* $\rho^+_\mu = 1$ *corresponds to an eigenvector* $x^{(\mu)}$*, whose support consists of all indices in the classes that have access to* μ*, and all vectors* x *of* $V_+(A, 1)$ *with* $\operatorname{supp} x = \operatorname{supp} x^{(\mu)}$ *are multiples of* $x^{(\mu)}$ *.*

(ii) $V_+(A, 1)$ *is generated by* $x^{(\mu)}$ *of* (i)*, for* μ *ranging over all spectral classes with* $\rho^+_\mu = 1$ *.*

(iii) $x^{(\mu)}$ *of* (i) *are extremals of* $V_+(A, 1)$ *. (Moreover,* $x^{(\mu)}$ *are linearly independent.)*

Note that the extremality and the **usual** linear independence of $x^{(\mu)}$ (involving linear combinations with possibly negative coefficients) can be deduced from the description of supports in part (i), and from the fact that in nonnegative algebra, spectral classes associated with the same ρ do not access each other. This linear independence also means that $V_+(A, 1)$ is a simplicial cone. See also [45, Th. 4.1].

Theorem 3.4.2. *[16, Th. 4.3.5] [50, Th. 2.8] Let* $A \in \mathbb{R}_+^{n\times n}$ *have* $\rho^\oplus(A) = 1$ *.*

(i) *Each component* $\widetilde{\mu}$ *of* $\mathcal{C}(A)$ *corresponds to an eigenvector* $x^{(\widetilde{\mu})}$ *defined as one of the columns* $A^*_{\cdot i}$ *with* $i \in N_{\widetilde{\mu}}$*, all columns with* $i \in N_{\widetilde{\mu}}$ *being multiples of each other.*

(ii) $V_\oplus(A, 1)$ *is generated by* $x^{(\widetilde{\mu})}$ *of* (i)*, for* $\widetilde{\mu}$ *ranging over all components of* $\mathcal{C}(A)$ *.*

(iii) $x^{(\widetilde{\mu})}$ *of* (i) *are extremals in* $V_\oplus(A, 1)$ *. (Moreover,* $x^{(\widetilde{\mu})}$ *are strongly linearly independent in the sense of [15].)*

Using this description of extremals we can now describe extremals of the core, since we know that $\mathrm{core}(A) = V^{\Sigma}(A^{\sigma_\Lambda})$ in both algebras, that the spectral classes of A^{σ_Λ} are derived from those of A, and that the access relations between classes of A^{σ_Λ} are similar to those between the classes of A (see Subsect. 2.4.).

Lemma 3.4.1. *For each $k \geqslant 1$, the set of extremals of $V^{\Sigma}(A^k)$ is the union of the sets of extremals of $V(A^k, \rho^k)$ for $\rho \in \Lambda(A)$.*

The following result describes extremals of the core in nonnegative algebra. It is not new (see the quotation). A vector $y \in \mathbb{R}^n_+$ is called normalized if $\max_i y_i = 1$ (but any other norm could be used as well).

Theorem 3.4.3. *[52, Theorem 4.7], [19]. Let $A \in \mathbb{R}^{n\times n}_+$.*

(i) *The set of extremals of $\mathrm{core}_+(A)$ is the union of the sets of extremals of $V_+(A^{\times\sigma}, \rho^\sigma)$ for all $\rho \in \Lambda_+(A)$, with $\sigma = \sigma_\rho$.*
(ii) *Each spectral class μ with cyclicity σ_μ corresponds to a set of distinct σ_μ normalized extremals of $\mathrm{core}_+(A)$, such that there exists an index in their support that belongs to μ, and each index in their support has access to μ.*
(iii) *Each set of extremals described in* (ii) *forms a simple cycle under the action of A.*
(iv) *There are no normalized extremals other than those described in* (ii). *The total number of normalized extremals equals the sum of cyclicities of all spectral classes of A.*

In [19] we obtained a similar description of extremals of the max-algebraic core.

Theorem 3.4.4. *[19]. Let $A \in \mathbb{R}^{n\times n}_+$.*

(i) *The set of extremals of $\mathrm{core}_\oplus(A)$ is the union of the sets of extremals of $V_\oplus(A^{\otimes\sigma}, \rho^\sigma)$ for all $\rho \in \Lambda(A)$, with $\sigma = \sigma_\rho$.*
(ii) *Each critical component $\widetilde{\mu}$ with cyclicity $\sigma_{\widetilde{\mu}}$ associated with some $\rho \in \Lambda_\oplus(A)$ corresponds to a set of distinct $\sigma_{\widetilde{\mu}}$ normalized extremals x of $\mathrm{core}_\oplus(A)$, which are (normalized) columns of $(A_\rho{}^{\otimes\sigma_\rho})^*$ with indices in $N_{\widetilde{\mu}}$.*
(iii) *Each set of extremals described in* (ii) *forms a simple cycle under the action of A.*
(iv) *There are no normalized extremals other than those described in* (ii). *The total number of normalized extremals equals the sum of cyclicities of all critical components of A.*

3.5. Ultimate periodicity and finite stabilization

In max algebra there are wide classes of matrices $A \in \mathbb{R}^{n\times n}_+$ where we have

$$\mathrm{span}_\oplus(A^{\otimes t}) = \mathrm{core}_\oplus(A)$$

for all big enough t. This is called the *finite stabilization of the core.* We list some special cases where this takes place.

- $\mathcal{S}_1$: **Irreducible matrices**.
- $\mathcal{S}_2$: **Ultimately periodic matrices.** This is when we have $A^{\otimes(t+\sigma)} = \rho^\sigma A^{\otimes t}$ for all sufficiently large t, with $\rho^\oplus(A)$. As shown in [43], this happens if and only if the Perron roots of all nontrivial classes of A equal $\rho^\oplus(A)$.
- $\mathcal{S}_3$: **Robust matrices.** For any nonzero vector $x \in \mathbb{R}^n_+$ the orbit $\{A^{\otimes t}x\}_{t\geqslant 1}$ hits an eigenvector of A, implying that the whole remaining part of the orbit consists of multiples of that eigenvector. The notion of robustness was introduced and studied in [17].
- $\mathcal{S}_4$: **Orbit periodic matrices:** For any nonzero vector $x \in \mathbb{R}^n_+$ the orbit $\{A^{\otimes t}\otimes x\}_{t\geqslant 1}$ hits an eigenvector of $A^{\otimes\sigma}$, implying that the remaining part of the orbit is periodic with some growth rate. See [49], Section 7 for characterization.
- $\mathcal{S}_5$: **Column periodic matrices.** This is when for any $i = 1,\dots,n$ we have $(A^{\otimes(t+\sigma)})_{\cdot i} = \rho_i^\sigma A^{\otimes t}_{\cdot i}$ for all large enough t and some ρ_i.

Observe that $\mathcal{S}_1 \subseteq \mathcal{S}_2 \subseteq \mathcal{S}_4 \subseteq \mathcal{S}_5$ and $\mathcal{S}_3 \subseteq \mathcal{S}_4$ (see, e.g., [19], Section 4). To see that $\operatorname{span}_\oplus(A^{\otimes t}) = \operatorname{core}_\oplus(A)$ for all large enough t in all these cases, observe that in the column periodic case ($\mathcal{S}_5$) all sequences of columns end up with periodically repeating eigenvectors of $A^{\otimes\sigma}$, which implies that $\operatorname{span}_\oplus(A^{\otimes t}) \subseteq \operatorname{core}_\oplus(A)$ for all large enough t, and hence also $\operatorname{span}_\oplus(A^{\otimes t}) = \operatorname{core}_\oplus(A)$. So finite stabilization of the core occurs in all these classes.

A necessary and sufficient condition for the finite stabilization can be formulated as follows.

Theorem 3.5.1. *[20]. Finite stabilization of* $\operatorname{core}_\oplus(A)$ *occurs if and only if all nontrivial classes of* A *are spectral.*

The action of A on its core in max algebra will be studied in more detail in [20].

References

[1] N. N. Vorob'ev, "The extremal matrix algebra", *Soviet Mathematics Doklady*, **4** (1963), 1220–1223.

[2] N. N. Vorob'ev, "Extremal Algebra of Positive Matrices", *Elektronische Informationsverarbeitung und Kybernetik*, **3**:1 (1967), 39–71 (In Russian).

[3] N. N. Vorob'ev, "Extremal Algebra of Non-Negative Matrices", *Elektronische Informationsverarbeitung und Kybernetik*, **6**:4–5 (1970), 303–312 (In Russian).

[4] P. I. Dudnikov, S. N. Samborskiĭ, "Endomorphisms of finitely generated free semimodules", *Math. USSR-Izv.*, **38**:1 (1992), 91–105.

[5] N. K. Krivulin, *Methods of Idempotent Algebra in Problems of Modeling and Analysis of Complex Systems*, Publ. St. Petersburg State. University, St. Petersburg, 2009 (In Russian).

[6] G. L. Litvinov, V. P. Maslov, A. N. Sobolevskiĭ, "Idempotent interval analysis and optimization problems", *Reliable Computing*, **7**:5 (2001), 353–377, arXiv: org/abs/math.SC/0101080.

[7] V. P. Maslov, “On new principle of superposition for optimization problems”, *Russian Math. Surveys*, **42**:3 (1987), 43–54.

[8] V. N. Kolokoltsov, V. P. Maslov, *Idempotent Analysis and Its Applications*, Kluwer Academic Pub., 1997.

[9] G. B. Shpiz, “An eigenvector existence theorem in idempotent analysis”, *Math. Notes*, **82**:3 (2007), 410–417.

[10] M. Akian, R. Bapat, S. Gaubert, “Max-plus algebras”, *Handbook of Linear Algebra*. V. 39: *Discrete Mathematics and its Applications*, eds. L. Hogben, R. Brualdi, A. Greenbaum, and R. Mathias, Chapman and Hall, London; New York, 2007.

[11] F. L. Baccelli, G. Cohen, G. J. Olsder, J. P. Quadrat, “Synchronization and Linearity: an Algebra for Discrete Event Systems”, *Journal of the Operational Research Society*, **45**:1 (1994).

[12] A. Berman, R .J. Plemmons, *Nonnegative Matrices in the Mathematical Sciences*, Society for Industrial and Applied Mathematics, Philadelphia, 1994.

[13] S. Bezuglyi, J. Kwiatkowski, K. Medynets, B. Solomyak, “Invariant measures on stationary Bratteli diagrams”, *Ergodic Theory Dynam. Systems*, **30**:4 (2010), 973–1007.

[14] R. A. Brualdi, H. J. Ryser, *Combinatorial Matrix Theory*, Cambridge Univ. Press, Cambridge, 1991.

[15] P. Butkovič, “Max-algebra: the linear algebra of combinatorics?”, *Linear Alg. Appl.*, **367** (2003), 313–335.

[16] P. Butkovič, *Max-linear Systems: Theory and Algorithms*, Springer, London, 2010.

[17] P. Butkovič, R. A. Cuninghame-Green, S. Gaubert, “Reducible spectral theory with applications to the robustness of matrices in max-algebra”, *SIAM J. on Matrix Analysis and Applications*, **31**:3 (2009), 1412–1431.

[18] P. Butkovič, H. Schneider, S. Sergeev, “Generators, extremals and bases of max cones”, *Linear Alg. Appl.*, **421** (2007), 394–406, arXiv: org/abs/math.RA/0604454.

[19] P. Butkovič, H. Schneider, S. Sergeev, “Two cores of a nonnegative matrix, submitted”, 2012, arXiv: org/abs/math.RA/1208.1939.

[20] P. Butkovič, H. Schneider, S. Sergeev, “On the max-algebraic core of a nonnegative matrix”, 2013, arXiv: org/abs/1305.7339.

[21] P. Butkovič, H. Schneider, S. Sergeev, “Z-matrix equations in max algebra, nonnegative linear algebra and other semirings”, *Linear and Multilin. Alg.*, **60**:10 (2012), 1191-1210, arXiv: org/abs/1110.4564.

[22] M.-H. Chi, “The long-run behaviour of Markov chains”, *Linear Alg. Appl.*, **244** (1996), 111–121.

[23] G. Cohen, D. Dubois, J. P. Quadrat, M. Viot, “Analyse du comportement périodique de systèmes de production par la théorie des dioïdes”, *Rapport de Recherche*, 1983, № 191.

[24] R. A. Cuninghame-Green, “Minimax Algebra”, *Lecture Notes in Economics and Mathematical Systems*. V. 166, Springer, Berlin, 1979.

[25] R. A. Cuninghame-Green, “Describing industrial processes with interference and approximating their steady-state behaviour”, *Operations Research*, **13**:1 (1962), 95–100.

[26] G. F. Frobenius, “Über Matrizen aus nicht negativen Elementen”, *Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften*, **23**, Sitzungsberichte, Berlin, 1912, 456–477.

[27] S. Gaubert, *Théorie des systèmes linéaires dans les dioïdes*, PhD thesis, Thèse de l’Ecole Nationale Supérieure des Mines de Paris, Juillet, 1992, 304 pp.

[28] S. Gaubert, R. D. Katz, “The Minkowski theorem for max-plus convex sets”, *Linear Alg. Appl.*, **421**:2-3 (2007), 356–369, arXiv: org/abs/math.GM/0605078.

[29] M. Gavalec, *Periodicity in Extremal Algebras*, Gaudeamus, Hradec Králové, 2004.

[30] M. Gondran, M. Minoux, "Valeurs propres et vecteurs propres dans les dioïdes", *EDF Bulletin de la Direction des Etudes et Recherches. Serie C. Mathematiques Informatique*, **2** (1977), 25–41.

[31] D. J. Hartfiel, *Nonhomogeneous Matrix Products*, World Scientific, Singapore, 2002.

[32] B. Heidergott, G.-J. Olsder, J. van der Woude, *Max-plus at Work*, Princeton Univ. Press, Princeton, 2005.

[33] R. D. Katz, H. Schneider, S. Sergeev, "On commuting matrices in max algebra and in classical nonnegative algebra", *Linear Alg. Appl.*, **436** (2012), 276–292, arXiv: org/abs/1005.1424.

[34] C.-K. Li, H. Schneider, "Applications of Perron-Frobenius theory to population dynamics", *J. of Mathematical Biology*, **44**:5 (2002), 450–462, arXiv: org/abs/math.RA/0109008.

[35] G. L. Litvinov, V. P. Maslov, "The correspondence principle for idempotent calculus and some computer applications", *Idempotency*, ed. J. Gunawardena, Cambridge Univ. Press, Cambridge, 1998, 420–443, arXiv: org/abs/math/0101021.

[36] G. L. Litvinov, V. P. Maslov, *Idempotent mathematics and mathematical physics*, **307**, AMS Contemporary Mathematics, Providence, 2005.

[37] G. L. Litvinov, V. P. Maslov, S. N. Sergeev, *Idempotent and tropical mathematics and problems of mathematical physics*. V. 1,2, Independent University of Moscow, Moscow, 2007, arXiv: org/abs/0709.4119.

[38] G. L. Litvinov, S. N. Sergeev, *Tropical and Idempotent Mathematics*, **495**, AMS Contemporary Mathematics, Providence, 2009.

[39] *Advances in Soviet Mathematics*. V. 13: *Idempotent analysis*, ed. V. P. Maslov, S. N. Samborskiĭ, AMS, Providence, 1992.

[40] W. M. McEneaney, *Max-plus Models for Nonlinear Control and Estimation*, Systems & Control: Foundations & Applications, Birkhäuser, Boston, 2006.

[41] G. Merlet, "Semigroup of matrices acting on the max-plus projective space", *Linear Alg. Appl.*, **432**:8 (2010), 1923–1935.

[42] M. Molnárová, "Generalized matrix period in max-plus algebra", *Linear Alg. Appl.*, **404** (2005), 345–366.

[43] M. Molnárová, J. Pribiš, "Matrix period in max-algebra", *Discrete Appl. Math.*, **103** (2000), 167–175.

[44] N. J. Pullman, "A geometric approach to the theory of nonnegative matrices", *Linear Alg. Appl.*, **4** (1971), 297–312.

[45] H. Schneider, "The influence of the marked reduced graph of a nonnegative matrix on the Jordan form and on related properties: A survey", *Linear Alg. Appl.*, **84** (1986), 161–189.

[46] B. De Schutter, "On the ultimate behaviour of the sequence of consecutive powers of a matrix in the max-plus algebra", *Linear Alg. Appl.*, **307** (2000), 103–117.

[47] E. Seneta, *Non-negative Matrices and Markov Chains*, Springer Series in Statistics, Springer, New York, 1981.

[48] S. Sergeev, "Max algebraic powers of irreducible matrices in the periodic regime: An application of cyclic classes", *Linear Alg. Appl.*, **431**:6 (2009), 1325–1339, arXiv: org/abs/0903.3960.

[49] S. Sergeev, H. Schneider, "CSR expansions of matrix powers in max algebra", *Transactions of AMS*, **364** (2012), 5969–5994, arXiv: org/abs/0912.2534.

[50] S. Sergeev, H. Schneider, P. Butkovič, "On visualization scaling, subeigenvectors and Kleene stars in max algebra", *Linear Alg. Appl.*, **431**:12 (2009), 2395–2406, arXiv: org/abs/0808.1992.

[51] G. Sierksma, "Limiting polytopes and periodic Markov chains", *Linear and Multilinear Algebra*, **46** (1999), 281–298.

[52] B.-S. Tam, H. Schneider, "On the core of a cone-preserving map", *Transactions of the AMS*, **343**:2 (1994), 479–524.

[53] E. Wagneur, “Moduloïds and pseudomodules. 1. Dimension theory”, *Discrete Math.*, **98** (1991), 57–73.

Список литературы

[1] Н. Н. Воробьев, “Экстремальная алгебра матриц”, *Докл. АН СССР*, **152**:1 (1963), 24–27.

[2] Н. Н. Воробьев, “Экстремальная алгебра положительных матриц”, *Elektronische Informationsverarbeitung und Kybernetik*, **3**:1 (1967), 39–71.

[3] Н. Н. Воробьев, “Экстремальная алгебра неотрицательных матриц”, *Elektronische Informationsverarbeitung und Kybernetik*, **6**:4–5 (1970), 303–312.

[4] П. И. Дудников, С. Н. Самборский, “Эндоморфизмы полумодулей над полукольцами с идемпотентной операцией”, *Изв. АН СССР. Сер. матем.*, **55**:1 (1991), 93–109.

[5] Н. К. Кривулин, *Методы идемпотентной алгебры в задачах моделирования и анализа сложных систем*, Изд. Санкт-Петербургского Гос. Ун-та, Санкт-Петербург, 2009.

[6] Г. Л. Литвинов, В. П. Маслов, А. Н. Соболевский, “Идемпотентная математика и интервальный анализ”, *Вычислительные технологии*, **6**:6 (2001), 47–70.

[7] В. П. Маслов, “О новом принципе суперпозиции для задач оптимизации”, *УМН*, **42**:3(255) (1987), 39–48.

[8] В. П. Маслов, В. Н. Колокольцов, *Идемпотентный анализ и его применение в оптимальном управлении*, Физматлит, М., 1994.

[9] Г. Б. Шпиз, “Теорема о существовании собственного вектора в идемпотентном анализе”, *Матем. заметки*, **82**:3 (2007), 459–468.

[10] M. Akian, R. Bapat, S. Gaubert, “Max-plus algebras”, *Handbook of Linear Algebra*. V. 39: *Discrete Mathematics and its Applications*, eds. L. Hogben, R. Brualdi, A. Greenbaum, and R. Mathias, Chapman and Hall, London; New York, 2007.

[11] F. L. Baccelli, G. Cohen, G. J. Olsder, J. P. Quadrat, “Synchronization and Linearity: an Algebra for Discrete Event Systems”, *Journal of the Operational Research Society*, **45**:1 (1994).

[12] A. Berman, R .J. Plemmons, *Nonnegative Matrices in the Mathematical Sciences*, Society for Industrial and Applied Mathematics, Philadelphia, 1994.

[13] S. Bezuglyi, J. Kwiatkowski, K. Medynets, B. Solomyak, “Invariant measures on stationary Bratteli diagrams”, *Ergodic Theory Dynam. Systems*, **30**:4 (2010), 973–1007.

[14] R. A. Brualdi, H. J. Ryser, *Combinatorial Matrix Theory*, Cambridge Univ. Press, Cambridge, 1991.

[15] P. Butkovič, “Max-algebra: the linear algebra of combinatorics?”, *Linear Alg. Appl.*, **367** (2003), 313–335.

[16] P. Butkovič, *Max-linear Systems: Theory and Algorithms*, Springer, London, 2010.

[17] P. Butkovič, R. A. Cuninghame-Green, S. Gaubert, “Reducible spectral theory with applications to the robustness of matrices in max-algebra”, *SIAM J. on Matrix Analysis and Applications*, **31**:3 (2009), 1412–1431.

[18] P. Butkovič, H. Schneider, S. Sergeev, “Generators, extremals and bases of max cones”, *Linear Alg. Appl.*, **421** (2007), 394–406, arXiv: org/abs/math.RA/0604454.

[19] P. Butkovič, H. Schneider, S. Sergeev, “Two cores of a nonnegative matrix, submitted”, 2012, arXiv: org/abs/math.RA/1208.1939.

[20] P. Butkovič, H. Schneider, S. Sergeev, “On the max-algebraic core of a nonnegative matrix”, 2013, arXiv: org/abs/1305.7339.

[21] P. Butkovič, H. Schneider, S. Sergeev, "Z-matrix equations in max algebra, nonnegative linear algebra and other semirings", *Linear and Multilin. Alg.*, **60**:10 (2012), 1191-1210, arXiv: org/abs/1110.4564.

[22] M.-H. Chi, "The long-run behaviour of Markov chains", *Linear Alg. Appl.*, **244** (1996), 111–121.

[23] G. Cohen, D. Dubois, J. P. Quadrat, M. Viot, "Analyse du comportement périodique de systèmes de production par la théorie des dioïdes", *Rapport de Recherche*, 1983, № 191.

[24] R. A. Cuninghame-Green, "Minimax Algebra", *Lecture Notes in Economics and Mathematical Systems*. V. 166, Springer, Berlin, 1979.

[25] R. A. Cuninghame-Green, "Describing industrial processes with interference and approximating their steady-state behaviour", *Operations Research*, **13**:1 (1962), 95–100.

[26] G. F. Frobenius, "Über Matrizen aus nicht negativen Elementen", *Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften*, **23**, Sitzungsberichte, Berlin, 1912, 456–477.

[27] S. Gaubert, *Théorie des systèmes linéaires dans les dioïdes*, PhD thesis, Thèse de l'Ecole Nationale Supérieure des Mines de Paris, Juillet, 1992, 304 pp.

[28] S. Gaubert, R. D. Katz, "The Minkowski theorem for max-plus convex sets", *Linear Alg. Appl.*, **421**:2-3 (2007), 356–369, arXiv: org/abs/math.GM/0605078.

[29] M. Gavalec, *Periodicity in Extremal Algebras*, Gaudeamus, Hradec Králové, 2004.

[30] M. Gondran, M. Minoux, "Valeurs propres et vecteurs propres dans les dioïdes", *EDF Bulletin de la Direction des Etudes et Recherches. Serie C. Mathematiques Informatique*, **2** (1977), 25–41.

[31] D. J. Hartfiel, *Nonhomogeneous Matrix Products*, World Scientific, Singapore, 2002.

[32] B. Heidergott, G.-J. Olsder, J. van der Woude, *Max-plus at Work*, Princeton Univ. Press, Princeton, 2005.

[33] R. D. Katz, H. Schneider, S. Sergeev, "On commuting matrices in max algebra and in classical nonnegative algebra", *Linear Alg. Appl.*, **436** (2012), 276–292, arXiv: org/abs/1005.1424.

[34] C.-K. Li, H. Schneider, "Applications of Perron-Frobenius theory to population dynamics", *J. of Mathematical Biology*, **44**:5 (2002), 450–462, arXiv: org/abs/math.RA/0109008.

[35] G. L. Litvinov, V. P. Maslov, "The correspondence principle for idempotent calculus and some computer applications", *Idempotency*, ed. J. Gunawardena, Cambridge Univ. Press, Cambridge, 1998, 420–443, arXiv: org/abs/math/0101021.

[36] G. L. Litvinov, V. P. Maslov, *Idempotent mathematics and mathematical physics*, **307**, AMS Contemporary Mathematics, Providence, 2005.

[37] G. L. Litvinov, V. P. Maslov, S. N. Sergeev, *Idempotent and tropical mathematics and problems of mathematical physics*. V. 1,2, Independent University of Moscow, Moscow, 2007, arXiv: org/abs/0709.4119.

[38] G. L. Litvinov, S. N. Sergeev, *Tropical and Idempotent Mathematics*, **495**, AMS Contemporary Mathematics, Providence, 2009.

[39] *Advances in Soviet Mathematics*. V. 13: *Idempotent analysis*, ed. V. P. Maslov, S. N. Samborskiĭ, AMS, Providence, 1992.

[40] W. M. McEneaney, *Max-plus Models for Nonlinear Control and Estimation*, Systems & Control: Foundations & Applications, Birkhäuser, Boston, 2006.

[41] G. Merlet, "Semigroup of matrices acting on the max-plus projective space", *Linear Alg. Appl.*, **432**:8 (2010), 1923–1935.

[42] M. Molnárová, "Generalized matrix period in max-plus algebra", *Linear Alg. Appl.*, **404** (2005), 345–366.

[43] M. Molnárová, J. Pribiš, "Matrix period in max-algebra", *Discrete Appl. Math.*, **103** (2000), 167–175.

[44] N. J. Pullman, "A geometric approach to the theory of nonnegative matrices", *Linear Alg. Appl.*, **4** (1971), 297–312.

[45] H. Schneider, "The influence of the marked reduced graph of a nonnegative matrix on the Jordan form and on related properties: A survey", *Linear Alg. Appl.*, **84** (1986), 161–189.

[46] B. De Schutter, "On the ultimate behaviour of the sequence of consecutive powers of a matrix in the max-plus algebra", *Linear Alg. Appl.*, **307** (2000), 103–117.

[47] E. Seneta, *Non-negative Matrices and Markov Chains*, Springer Series in Statistics, Springer, New York, 1981.

[48] S. Sergeev, "Max algebraic powers of irreducible matrices in the periodic regime: An application of cyclic classes", *Linear Alg. Appl.*, **431**:6 (2009), 1325–1339, arXiv: org/abs/0903.3960.

[49] S. Sergeev, H. Schneider, "CSR expansions of matrix powers in max algebra", *Transactions of AMS*, **364** (2012), 5969–5994, arXiv: org/abs/0912.2534.

[50] S. Sergeev, H. Schneider, P. Butkovič, "On visualization scaling, subeigenvectors and Kleene stars in max algebra", *Linear Alg. Appl.*, **431**:12 (2009), 2395–2406, arXiv: org/abs/0808.1992.

[51] G. Sierksma, "Limiting polytopes and periodic Markov chains", *Linear and Multilinear Algebra*, **46** (1999), 281–298.

[52] B.-S. Tam, H. Schneider, "On the core of a cone-preserving map", *Transactions of the AMS*, **343**:2 (1994), 479–524.

[53] E. Wagneur, "Moduloïds and pseudomodules. 1. Dimension theory", *Discrete Math.*, **98** (1991), 57–73.

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There is no conflict of interests.

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Received 21 June 2019
Reviewed 29 July 2019
Accepted for press 23 August 2019

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Конфликт интересов отсутствует.

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Поступила в редакцию 21 июня 2019 г.
Поступила после рецензирования 29 июля 2019 г.
Принята к публикации 23 августа 2019 г.